LESSON PLAN: CHAIN RULE

The chain rule is used extremely often, and is a concept you can't do without.

If it's so important, why? What is it, exactly? Before I get to the teaching, I need to warn you. This is one of the most common things that people forget to do. The chain rule is needed every single time you take a derivative. I'll show you why I haven't mentioned it until now, but from now on, you need to consider the effects of the chain rule EVERY time you take a derivative. Up until now, you haven't been able to take the derivative of anything complicated. I've only given you things that are fairly simple. I'm sorry to tell you that things can get much, much harder! Here are a couple of examples, to show you what I mean.

$$\sin^4(16x^3 - \cos x)$$

$$(x^2-4)^3(16-x^3)$$

$$\frac{\sin^3(x^2-5)}{(12x-55)^2}$$

Before now, you wouldn't be able to take the derivative of the functions above. They are too complex, and require the chain rule. With some problems, the effects of the chain rule are negligible, and those are the functions we've been working with thus far.

What's the chain rule?

First, I'll give you the formal definition. I know it is going to not make much sense, but I'll thoroughly explain it afterward.

$$y = f(g(x))$$

$$y' = f'(g(x)) \times g'(x)$$

If you don't understand that, don't worry. Nobody really expects you to at first. So, what the heck does that mean? Simply, this means when you have one function inside of another. Here's an example:

$$y = (3x+2)^4$$

The inside function is $3x+2$, and the outside function is $(\text{something})^4$. One function is inside of the other. That's the best way to think about a complicated function. You should think of it in parts, like I just showed. Now, how do we take the derivative of this? To begin, you pretend that the inside of the function is just a simple x, and you don't do anything with it. You leave it alone completely. For this part, pretend that we are taking the derivative of $(\text{something})^4$. It doesn't matter what the "something" is. Just leave it alone, and take the derivative as if the "something" is just an "x".

$$y' = 4 \times (\text{something})^3$$

That is exactly what I just told you to do, however the problem isn't complete. Next, you must multiply that answer by the derivative of the "something". In the first part, we completely ignore what the "something" is. Now, you look at ONLY the "something", and take the derivative of that. This is going to be our final answer, unless we can simplify something.

$$
\begin{aligned}
y' &= 4 \times (3x+2)^3 \times \frac{\mathrm{d}(3x+2)}{\mathrm{d}x} \\
&= 4(3x+2)^3(3+0) \\
&= 12(3x+2)^3
\end{aligned}
$$

As you can see, I didn't leave the word "something" in the middle of my answer. I just left it alone completely in the first part. I didn't touch it. I just rewrote it. I'm being deliberately redundant here. I want to say this as many ways as possible, so that you completely understand what I am saying. This is extremely important, and I would venture to guess that it is the most common mistake made in calculus. Everyone messes up the chain rule at some point.

This is an extremely fundamental topic in calculus. This rule should be applied to every single derivative that you are taking. Sometimes, it will not affect your answer, but most of the time it will. If you forget to use the chain rule, you will get the answer wrong! This is a rule. This is how you take derivatives. It is as important as the order of operations. It is needed. This is how you take a derivative properly. You need to practice this thoroughly!

Before I get to another example for the chain rule, I want to show you why the chain rule was not needed for all the derivatives we've been taking until now. The reason is simply that every time we take the derivative, the inside function has just been an "x". For example,

$$y = \sin x$$

The outside function is "sin", and the inside is just "x". What happens if we use the chain rule here?

$$
\begin{aligned}
y' &= \cos(\text{something}) \times \frac{\mathrm{d}(\text{something})}{\mathrm{d}x} \\
&= \cos x \times \frac{\mathrm{d}(x)}{\mathrm{d}x} \\
&= \cos x \times 1 \\
&= \cos x
\end{aligned}
$$

The only difference from what we used to do is when we multiply the derivative of the inside "something". The "something" has always been just an "x". The derivative of x is just 1, and when you multiply by 1 nothing changes! The chain rule, the part where we multiply by the derivative of the inside, does absolutely nothing. It is completely negligible. That is what we've never needed it! However, now that you know about the chain rule, you should always use it. Even when it isn't going to change the answer. You need to make it a habit, or you will forget to do it at a time where you really need to. It's inevitable that you will mess this up a few times, but hopefully not much more than that.

Remember this:

To take a derivative, you look at the most "outside" function. In most cases, it will be an exponent on parenthesis. You pretend that everything on the inside is just an x. It could be absolutely anything, and it doesn't matter. Take the derivative as usual, but leave the inside completely untouched. Just copy it from one line to the next. After you do that, you must multiply that answer by the derivative of the inside stuff. The same stuff you previously skipped over and ignored. That is the most simple way to describe and understand the chain rule.

Want to try another example?

$$y = \sin(10x + 5)$$

The most outside part is the "sin", so start there. Take the derivative, and then move to the inside part.

$$\begin{aligned} y' &= \cos(10x + 5) \times (10 + 0) \\ &= 10\cos(10x + 5) \end{aligned}$$

See how that works?

One more basic problem, before I show you some hard ones?

$$\begin{aligned} y &= 5(x^3 + 7)^3 \\ &= 5(\text{something})^3 \end{aligned}$$

$$\begin{aligned} y' &= 15(\text{something})^2 \times \frac{\mathrm{d}(\text{something})}{\mathrm{d}x} \\ &= 15(x^3 + 7)^2 \times (3x^2 + 0) \\ &= 45x^2(x^3 + 7)^2 \end{aligned}$$

I have two things to admit to you. First of all, you don't have to actually write "something", like I've been doing in every problem. That is just the way I suggest you think about these problems. Visualize that step as you are using the chain rule, but don't write it down. Second, you really don't need to show the step where I write the symbol for the derivative. You can just take the derivative. If you look in this last example, you should skip the entire first step. You can go directly to the second step that I've shown. I'm just showing extra so that you understand it a little bit better.

I have two more examples for you. These are a bit more challenging. If you can do this type of problem on your own, you will be able to do any chain rule problem. That said, I suggest you try doing the following two examples yourself. Don't look at my solution, and work it out on your own. Only scroll further to see my answer when you get stuck. That is the best way for you to learn! (I promise)

$$y = (x+3)^2(2x^2 - 5x)$$

Recall the product rule:

$$y' = f'(x) \times g(x) + g'(x) \times f(x)$$

Here,

$$f(x) = (x+3)^2$$
$$f'(x) = 2(x+3)^1 \times (1)$$
$$= 2x + 6$$
$$g(x) = 2x^2 - 5x$$
$$g'(x) = 4x - 5$$

$$y' = (2x+6)(2x^2-5x) + (4x+5)(x+3)^2$$

These can get pretty complicated, but I hope you could follow me! This answer is technically correct, however many teachers will want you to go further than this. When you have to use both the product rule and chain rule, often times you can factor and simplify things a great deal. I'll show you what to do with this answer.

$$y' = (x+3)\Big[2(2x^2-5x) + (4x+5)(x+3)\Big]$$
$$= (x+3)\Big[4x^2 - 10x + 4x^2 + 12x + 5x + 15\Big]$$
$$= (x+3)\Big[8x^2 + 7x + 15\Big]$$

See how much nicer that looks? I think so at least, and many of your teachers will require you to do this simplification! Be on alert whenever you are using the product rule to take the derivative of something complicated.

For my last example, I will give you my favorite challenge problem. Whenever I tutor calculus I, I always ask this type of function. Like I said before, try to do it without my help. If you can, that is extremely impressive! This is like the test of all tests. If you can figure this one out, consider yourself an expert of the chain rule! I'm going to break it down step by step, and explain why it needs to be done this way.

$$y = 3\cos^3(3x^2 - 15)$$

The way you should always start taking a derivative is by asking yourself, "What is the most outside part?" This is where we always begin. If you remember, you need to ignore the part on the inside, and take the derivative of only the outermost part of the function. Work your way from the outside to the inside. So, what's the answer? What is the outermost part of this function? What should we start taking the derivative of?

To be honest, I'm being sneaky. Most of you would say, "cosine of course!" That is exactly what I expect you to say, however that is wrong! Let me rewrite the problem. If you don't know what I am talking about, you will in just a second.

$$y = 3\big(\cos(3x^2 - 15)\big)^3$$

The reason this may have tricked you, is that I placed the exponent right next to the "cos". This is a very common way to write exponents on trigonometric functions, but it always leads to confusion! If you get a problem with a trig function that has a power on it, rewrite it like I just did. Be careful though, the parenthesis around the "cos" are extremely important. Otherwise, the exponent isn't in the correct place, and you will confuse yourself even further.

Now, I'll ask you one more time. What is the most outside part? Of course, you can now see the most outside part is the exponent. For the first part, you should ignore everything inside the parenthesis with the exponent on it. I'm going to show you the full solution now, and then give a brief explanation afterward.

$$\begin{aligned} y' &= 9\big(\cos(3x^2 - 15)\big)^2 \times \frac{\mathrm{d}\big(\cos(3x^2 - 15)\big)}{\mathrm{d}x} \\ &= 9\big(\cos(3x^2 - 15)\big)^2 \times \big(-\sin(3x^2 - 15)\big) \times \frac{\mathrm{d}(3x^2 - 15)}{\mathrm{d}x} \\ &= 9\big(\cos(3x^2 - 15)\big)^2 \times \big(-\sin(3x^2 - 15)\big) \times (6x) \\ &= -54\cos^2(3x^2 - 15)\sin(3x^2 - 15) \end{aligned}$$

There was a chain rule needed inside of a chain rule! Isn't that really neat? Basically, you work your way in from the most outside piece of the function. Then you take one step inside. Then another step inside. Until you finally reach the innermost basic function that was trapped inside everything else. As you go along, you need to respect the chain rule, and multiply by the derivative of the respective next inside piece of the function. You multiply by that derivative. Then you need to determine if you will need to use the chain rule on the that derivative as well. You just keep multiplying them!

Do you see how I simplified my answer? Make sure you bring all of the coefficients to the very front of the expression. Thats is very important for style points, and you might get points off on a test if you don't do that!

That was my last example. I really like teaching that one. These problems can get pretty complicated, and I suggest you practice the chain rule. It needs to be second nature, and a habit for every time you take a derivative. Always write it! This is the most common mistake in calculus. Don't forget the chain rule! Remind yourself before you even begin a problem.